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The Thermo-field Dynamics Method for Electron with Two-mode Electromagnetic Field

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Abstract: The quick progress in quantum entanglement research allows us not one to study quantum systems down to N-bodies but also to take a new look at these systems in different branches of physics; particularly the statistical thermodynamics where the application of the thermo-field dynamics (TFD) method to the investigation of entanglement is fruitful. Because the traditional methods based on the identification of a specific parameter show their limit. The process using (TFD) facilitates the understanding of entanglement because it focuses directly on the eigenstate of the system and it is useful in the equilibrium and the non-equilibrium states also. In this context, the (TFD) method is used in this paper to analyze entanglement of an electron interacting with a two-mode electromagnetic field assimilated to an electron with two harmonic oscillators. Entanglement entropies are derived between concerned, not concerned harmonic oscillator and electron compute when the system is in a thermodynamic equilibrium and non-equilibrium state. For the equilibrium case, an increase in the number of particles per unit volume increases the quantum entanglement consequently entanglement appears more important for the couple oscillator-electron than the one electron, this trend is reversed for the non-equilibrium case. By respecting the entanglement parameters, such results allow us to know the relative equilibrium state of the overall system.

Keywords: Entanglement, Equilibrium and Non-equilibrium Thermodynamic State, Electron-two Harmonic Oscillators

1. Introduction

In recent years, entanglement measurements have proven to be effective tools to characterize and understand quantum processes. In particular: quantum cryptography [1, 2], quantum teleportation [3], quantum computing [4, 5]... etc. From the one side, given the recent progress in the development of the study of quantum systems, the concept of quantum information is focused on the measurement of entanglement entropies defined via bipartite systems [6-12]. We mention herein: the entangled photons. The quantum source of entangled photons become experimentally accessible when atom interact with a strong electromagnetic field and a high power laser. It is considered an important phenomenon in modern physics, particularly in quantum optics [13-15], quantum information [16, 17]... etc. For example refs [18, 19] have studied the electron-hole pair in a semiconductor as a source of entangled photons. On the other side and recently ref [20] treated a new approach to examine quantum entanglement in Hilbert space using TFD method. Subsequently, this method was developed by ref [21] for the coupled harmonic oscillators system.

In this paper, entanglement entropies are studied analytically and numerically on the basis of TFD for the system of entangled photons in a two-mode electromagnetic field to allow access on some subtle and universal features.

2. Problem Formulation: Eigenenergy Solution

We are interested in this section to study a Hamiltonian of an electron in interaction with a two-mode electromagnetic field. It can be defined as follows

$$\hat{H} = \frac{1}{2m_e}\left(-i\hbar\frac{\partial}{\partial r} + \frac{e}{c}\hat{A}\right)^2 + \frac{\hbar\omega_1}{2}\left(q_1^2 - \frac{\partial^2}{\partial q_1^2}\right) + \frac{\hbar\omega_2}{2}\left(q_2^2 - \frac{\partial^2}{\partial q_2^2}\right) - eU\left(r\right), \tag{1}$$

where m_e and e are respectively the mass and the charge of the electron and

$$\hat{A} = \sum_j \sqrt{\frac{2\pi c^2\hbar}{\omega_j V_j}}\{\nu_j \exp\left(i\left(\omega_j - ik_j r\right)\right)a_j + \nu_j^* \exp\left(-i\left(\omega_j - ik_j r\right)\right)a_j^+\}, \tag{2}$$

is the vector potential. It is related with the field variables by the relation

$$\hat{A} = s_1\nu_1 q_1 + s_2\nu_2 q_2. \tag{3}$$

We can identify ν_j as polarisation of the wave vector k, $s_j = \sqrt{\frac{4\pi c^2}{\omega_j V_j}}$ where $(j = 1.2)$ and V_j are the quantization volume. Here $U(r)$ is the atomic potential and is neglected on the assumption that the electromagnetic field is considered strong. Consider $m_e = 1$ and $\hat{A}$ in (2.3) is polarized in the direction of (xoz) plane. As a consequence, (2.1) becomes

$$\begin{aligned}\hat{H} =& \frac{1}{2}\big(-i\hbar\frac{\partial}{\partial x} + \lambda_1\nu_1 q_1 + \lambda_2\nu_2 q_2\big)^2 + \frac{1}{2}\big(-i\hbar\frac{\partial}{\partial y}\big)^2 + \frac{1}{2}\big(-i\hbar\frac{\partial}{\partial z} + \lambda_1\nu_1 q_1 + \lambda_2\nu_2 q_2\big)^2 \\ &+ \frac{\hbar\omega_1}{2}\big(q_1^2 - \frac{\partial^2}{\partial q_1^2}\big) + \frac{\hbar\omega_2}{2}\big(q_2^2 - \frac{\partial^2}{\partial q_2^2}\big).\end{aligned} \tag{4}$$

Note that $\lambda_1 = \sqrt{\frac{4\pi}{\omega_1 V_1}}$ and $\lambda_2 = \sqrt{\frac{4\pi}{\omega_2 V_2}}$. We perform a unitary transformation following the variables q_1, q_2 through expression:

$$\hat{\Gamma} = \exp\left[\eta q_1\frac{\partial}{\partial q_2} - \chi q_2\frac{\partial}{\partial q_1}\right], \tag{5}$$

such as the product $q_1 q_2$ vanishes. Hamiltonian (2.4) provides a new form denoted by

$$\hat{H} = -\frac{\hbar^2}{2}\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\big) + \frac{\hbar}{2}\varpi_1^2 q_1^2 + \frac{\hbar}{2}\varpi_2^2 q_2^2 - i\hbar\theta_1\hat{q}_1\big(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\big) - i\hbar\theta_2 q_2\big(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\big) - \frac{\gamma_1^2}{2}\frac{\partial^2}{\partial q_1^2} - \frac{\gamma_2^2}{2}\frac{\partial^2}{\partial q_2^2}, \tag{6}$$

where

$$\varpi_1^2 = 2\hbar\left[\frac{\omega_2}{\omega_1}\left(\omega_2 + \lambda_2^2\right)\left(1 + \left(\frac{\mu}{\sigma}\right)^2\right) - \sqrt{\mu^2 + \sigma^2}\left(\frac{\frac{\omega_2}{\omega_1}\left(\omega_2 + \lambda_2^2\right)^2 - \left(\omega_1 + \lambda_1^2\right)\left(\omega_2 + \lambda_2^2\right)}{4\lambda_1^2\lambda_2^2} + \frac{1}{2}\right)\right], \tag{7}$$

$$\varpi_2^2 = \frac{\hbar}{2}\left[\left(\omega_2 + \lambda_2^2\right) + \frac{1}{\sqrt{\mu^2 + \sigma^2}}\left[\mu\left(\omega_2 + \lambda_2^2\right) + \frac{1}{2}\frac{\omega_1}{\omega_2}\sigma^2\right]\right], \tag{8}$$

$$\theta_1 = \hbar\left[\lambda_1\nu_1 - \lambda_2\sqrt{\frac{\omega_2}{\omega_1}}\frac{\sqrt{\mu^2 + \sigma^2} - \mu}{\sigma}\nu_2\right], \tag{9}$$

$$\theta_2 = \frac{\hbar}{2}\left[\frac{1}{\sqrt{\mu^2 + \sigma^2}}\left(\sqrt{\frac{\omega_1}{\omega_2}}\sigma\lambda_1\nu_1 + \mu\lambda_2\nu_2\right) + \lambda_2\nu_2\right], \tag{10}$$

$$\gamma_1^2 = \frac{\hbar}{2}\omega_1\left(1 + \frac{\mu}{\sqrt{\mu^2 + \sigma^2}}\right), \tag{11}$$

$$\gamma_2^2 = 2\hbar\omega_2\left[1 + \left(\frac{\mu}{\sigma}\right)^2\left(1 - \frac{\sqrt{\mu^2 + \sigma^2}}{\mu}\right)\right], \tag{12}$$

$$\mu = \frac{\omega_2}{\omega_1}\left(\omega_2 + \lambda_2^2\right) - \left(\omega_1 + \lambda_1^2\right), \tag{13}$$

and

$$\sigma = 2\sqrt{\frac{\omega_2}{\omega_1}}\lambda_1\lambda_2\nu_1\nu_2. \tag{14}$$

To diagonalize (2.6), we introduced a second unitary operator as

$$\hat{O} = \exp\left(i\kappa_1\frac{\partial}{\partial z}\frac{\partial}{\partial q_2} + i\varepsilon_2\frac{\partial}{\partial z}\frac{\partial}{\partial q_1} + i\varepsilon_1\frac{\partial}{\partial x}\frac{\partial}{\partial q_1} - i\kappa_2 x\frac{\partial}{\partial z}\right). \tag{15}$$

Consequently, when we perform the appropriate calculation we obtain

$$\hat{H} = -\frac{\hbar^2}{2}\left(\left(1-S\right)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(1-Q\right)\frac{\partial^2}{\partial z^2}\right) + \frac{\hbar}{2}\varpi_1^2 q_1^2 + \frac{\hbar}{2}\varpi_2^2 q_2^2 - \frac{\gamma_1^2}{2}\frac{\partial^2}{\partial q_1^2} - \frac{\gamma_2^2}{2}\frac{\partial^2}{\partial q_2^2}, \tag{16}$$

where

$$\kappa_1 = \frac{\theta_2}{\hbar\varpi_2^2}\left(1 + i\hbar\kappa_2\right), \quad |\kappa_2| = \frac{\theta_2^2}{\hbar\varpi_2^2\left(1 - \frac{\theta_2^2}{\hbar\varpi_2^2}\right)}, \tag{17}$$

$$\varepsilon_1 = \frac{\theta_1}{\hbar\varpi_1^2}, \quad \varepsilon_2 = \varepsilon_1\left(1 + \frac{\theta_2^2}{\hbar\varpi_2^2\left(1 - \frac{\theta_2^2}{\hbar\varpi_2^2}\right)}\right), \tag{18}$$

$$S = \theta_1\varepsilon_1, \quad \text{and } Q = \frac{\theta_1^2}{\hbar\varpi_1^2}\left(1 + 2i\kappa_2\hbar\right) - \left(\frac{\theta_1^2}{\hbar\varpi_1^2} - 1\right)\hbar^2\kappa_2^2. \tag{19}$$

Expression (2.16) shows well the correlation between particles. It is impossible to study individually from where: result of entanglement. This model is considered in ref [22] to analyse entanglement by solving the Schrödinger equation and on the basis of the Schmidt decomposition, specifying the case of the photon with a single mode electromagnetic field is exposed in ref [23].

As a result from (2.16), the corresponding eigenenergy is

$$E_{k,m,n} = \frac{\hbar^2}{2}k_y^2 + \frac{\hbar^2}{2}\left(1-Q\right)k_z^2 + \frac{\hbar^2}{2}\left(1-S\right)k_x^2 + \hbar\gamma_1\varpi_1\left(n_1 + \frac{1}{2}\right) + \hbar\gamma_2\varpi_2\left(n_2 + \frac{1}{2}\right) \tag{20}$$

3. System at Equilibrium State

From the Fock space, the system is described as the eigenstate $|n_1, n_2; 1_{k_3}\rangle$ and the following eigenvalue equation

$$H|n_1, n_2; 1_{k_3}\rangle = E_{n_1,n_2,k_3}|n_1, n_2; 1_{k_3}\rangle, \tag{21}$$

$|n_1, n_2\rangle$ and $|1_{k_3}\rangle$ are respectively the states of the field and the electron in the atom. The global state is thus described as a triple sum: two harmonic oscillators and electron in atom. It is represented through a continuous non-separable variable k_3 so Hamiltonian (2.16) can be re-expressed as follows

$$\begin{aligned}\hat{H} = \sum_{n_1,n_2=0}^{\infty}\int d^3k_3\Bigg[&\frac{\hbar^2}{2}\left(k_y^2 + \left(1-Q\right)k_z^2 + \left(1-S\right)k_x^2\right)\\ &+ \hbar\gamma_1\varpi_1\left(n_1 + \frac{1}{2}\right) + \hbar\gamma_2\varpi_2\left(n_2 + \frac{1}{2}\right)\Bigg]|n_1, n_2; 1_{k_3}\rangle\langle n_1, n_2; 1_{k_3}|\end{aligned} \tag{22}$$

TFD method is defined as a direct application of the system state based on the formula of thermo-statistical physics. To start, we define the partition function as

$$Z = Tr_{1,2,3}e^{-\beta\hat{H}}. \tag{23}$$

It is given following (3.2) by

$$Z\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)=\left(\frac{2\pi}{\beta\hbar^2}\right)^{\frac{3}{2}}\frac{1}{(1-Q)^{\frac{1}{2}}}\frac{1}{(1-S)^{\frac{1}{2}}}\frac{\exp\left(\frac{\beta\hbar\gamma_1}{2}\varpi_1\right)\exp\left(\frac{\beta\hbar\gamma_2}{2}\varpi_2\right)}{\left(1-\exp\left(-\beta\hbar\gamma_1\varpi_1\right)\right)\left(1-\exp\left(-\beta\hbar\gamma_2\varpi_2\right)\right)}. \tag{24}$$

Here β is the inverse of the temperature. The ordinary density matrix reads:

$$\begin{aligned}\rho_{eq}\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)=&\frac{e^{-\beta\hat{H}}}{Z\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)}\\=&\frac{1}{Z\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int d^3k_3\exp\Big[-\frac{\beta\hbar^2}{2}\big(k_y^2+\big(1-Q\big)k_z^2+\big(1-S\big)k_x^2\big)\\&-\beta\hbar\gamma_1\varpi_1\big(n_1+\frac{1}{2}\big)-\beta\hbar\gamma_2\varpi_2\big(n_2+\frac{1}{2}\big)\Big]|n_1,n_2;1_{k_3}\rangle\langle n_1,n_2;1_{k_3}|.\end{aligned} \tag{25}$$

In relation to (3.5), the statistical eigenvector $|\psi\rangle$ is expressed as

$$\begin{aligned}|\psi\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)\rangle=&\sum_{(n_1,n_2)=0}^{\infty}\int d^3k_3\sqrt{\rho_{eq}\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)}|n_1,n_2;1_{k_3}\rangle|\tilde{n}_1,\tilde{n}_2;\tilde{1}_{k_3}\rangle\\=&\frac{1}{\sqrt{Z\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)}}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int d^3k_3\exp\Big[-\frac{\beta\hbar^2}{4}\Big(k_y^2+\big(1-Q\big)k_z^2+\big(1-S\big)k_x^2\Big)\\&-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big(n_1+\frac{1}{2}\big)-\frac{\beta\hbar\gamma_2}{2}\varpi_1\big(n_2+\frac{1}{2}\big)\Big]|n_1,n_2;1_{k_3}\rangle|\tilde{n}_1,\tilde{n}_2;\tilde{1}_{k_3}\rangle.\end{aligned} \tag{26}$$

The extended density matrix define through the product of the eigenvector (3.6) and their conjugate follow the expression

$$\hat{\rho}=|\psi\rangle\psi|, \tag{27}$$

will be an essential tool to describe quantum entanglement in TFD so it is given by

$$\begin{aligned}\rho_{(1,2,3)}\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)=&\frac{1}{Z\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)}\sum_{(m_1,m_2)=0}^{\infty}\sum_{(n_1,n_2)=0}^{\infty}\int d^3k_3\int d^3k_3^{'}\\&\times\exp\Big[-\frac{\beta\hbar^2}{4}\Big(\big(k_{3y}^2+k_{3y}^{'2}\big)+\big(1-Q\big)\big(k_{3z}^2+k_{3z}^{'2}\big)+\big(1-S\big)\big(k_{3x}^2+k_{3x}^{'2}\big)\Big)\\&-\frac{\beta\hbar\gamma_1\varpi_1}{2}\big(m_1+n_1\big)-\frac{\beta\hbar\gamma_2\varpi_2}{2}\big(m_2+n_2\big)-\frac{\beta\hbar\gamma_1}{2}\varpi_1-\frac{\beta\hbar\gamma_2}{2}\varpi_2\Big]\\&\times|m_1,m_2;1_{k_3}\rangle\langle n_1,n_2;1_{k_3^{'}}||\tilde{m}_1,\tilde{m}_2;\tilde{1}_{k_3}\rangle\langle\tilde{n}_1,\tilde{n}_2;\tilde{1}_{k_3^{'}}|.\end{aligned} \tag{28}$$

Investigation of entanglement in multiparticles is discussed in refs [24, 25]. At this level, we have all the means to study entanglement of the system on statistical thermodynamics properties.

3.1. Entanglement Between the Couple {1,2} Oscillators and Electron {3}

To understand the overall situation, consider the couple $\{1,2\}$ as not concerned and we examine the state $\{3\}$ of the electron as the reduced state. Their expression is provided from

$$\begin{aligned}\rho_3\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)=&\mathrm{Tr}_{1,2}\rho_{(1,2,3)}\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)\\=&\frac{\exp\left(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\right)\exp\left(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\right)}{Z\left(\varepsilon_1,\varepsilon_2,K\right)}\\&\times\int d^3k_3\int d^3k_3^{'}\frac{1}{\left(1-\exp\left(-\beta\hbar\gamma_1\varpi_1\right)\right)}\frac{1}{\left(1-\exp\left(-\hbar\beta\gamma_2\varpi_2\right)\right)}\end{aligned}$$

$$\times \exp\left[-\frac{\beta\hbar^2}{4}\left(\left(k_{3y}^2+k_{3y}'^2\right)+(1-Q)\left(k_{3z}^2+k_{3z}'^2\right)+(1-S)\left(k_{3x}^2+k_{3x}'^2\right)\right)\right] \\ \times |1_{k_3}\rangle\langle 1_{k_3'}||\tilde{1}_{k_3}\rangle\langle\tilde{1}_{k_3'}|. \tag{29}$$

The traditionally entanglement entropy is described as

$$S_3 = -k_B \mathrm{Tr}_3\left[\rho_3 \log \rho_3\right]. \tag{30}$$

The calculation yields

$$\hat{S}_3 = k_B 2^3 \times \frac{\left(\frac{2\pi}{\beta\hbar^2}\right)^{\frac{3}{2}}}{\sqrt{1-Q}\sqrt{1-S}}\left(6\frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}} - \log\left(\frac{\sqrt{1-Q}\sqrt{1-S}}{\left(\frac{2\pi}{\beta\hbar^2}\right)^{\frac{3}{2}}}\right)\right). \tag{31}$$

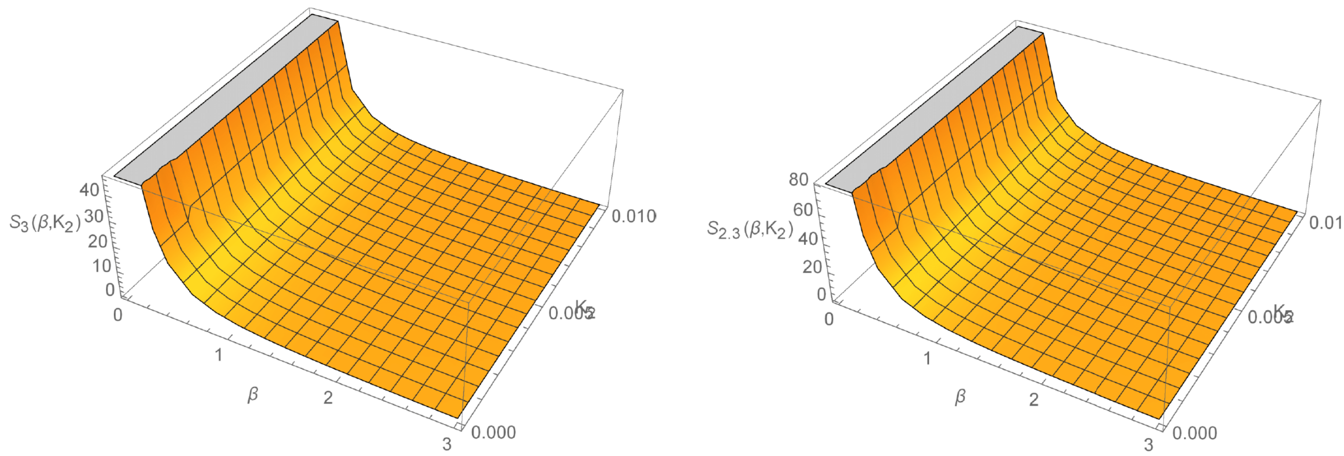

Figure 1. *Results of entanglement entropies $S_3(\beta,\kappa_2)$ in (3.11) and $S_{2.3}(\beta,\kappa_2)$ in (3.14).*

3.2. Entanglement Between Oscillator {1} and the Couple {2,3} Oscillator-electron

Consider a harmonic oscillator $\{1\}$ is not concerned and we study the $\{2,3\}$ state as the reduced state. We have

$$\begin{aligned}\rho_{2,3}\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right) =& \mathrm{Tr}_1\rho_{(1,2,3)}\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)\\ &= \frac{\exp\left(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\right)\exp\left(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\right)}{Z\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)}\frac{1}{\left(1-\exp\left(-\beta\hbar\gamma_1\varpi_1\right)\right)}\\ &\times\sum_{(m_2,n_2)=0}^{\infty}\int d^3k_3\int d^3k_3'\exp\left[-\frac{\beta\hbar\gamma_2}{2}\varpi_2(m_2+n_2)\right]\\ &\times\exp\left[-\frac{\beta\hbar^2}{4}\left(\left(k_{3y}^2+k_{3y}'^2\right)+(1-Q)\left(k_{3z}^2+k_{3z}'^2\right)+(1-S)\left(k_{3x}^2+k_{3x}'^2\right)\right)\right]\\ &\times|m_2;1_{k_3}\rangle\langle n_2;1_{k_3'}||\tilde{m}_2;\tilde{1}_{k_3}\rangle\langle\tilde{n}_2;\tilde{1}_{k_3'}|.\end{aligned} \tag{32}$$

In this manner we obtain the entanglement entropy

$$S_{2,3} = -k_B \mathrm{Tr}_{2,3}\left[\rho_{2,3}\log\rho_{2,3}\right], \tag{33}$$

as

$$\begin{aligned}\hat{S}_{2,3} =& k_B 2^3 \times \frac{\left(\frac{2\pi}{\beta\hbar^2}\right)^{\frac{3}{2}}}{\sqrt{1-Q}\sqrt{1-S}} \frac{\left(\exp\left(\beta\hbar\gamma_2\varpi_2\right)-1\right)}{\left(\exp\left(\frac{\beta\hbar\gamma_2\varpi_2}{2}\right)-1\right)^2} \left(6\frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}} + \frac{\beta\hbar\gamma_2\varpi_2}{\left(\exp\left(\beta\hbar\gamma_2\varpi_2\right)-1\right)} \right. \\ & \left. - \log\left(\frac{\sqrt{1-Q}\sqrt{1-S}}{\left(\frac{2\pi}{\beta\hbar^2}\right)^{\frac{3}{2}}} \frac{\left(\exp\left(\beta\hbar\gamma_2\varpi_2\right)-1\right)}{\exp\left(\beta\hbar\gamma_2\varpi_2\right)} \right) \right).\end{aligned} \tag{34}$$

From the numerical part we set: $V_1 = 2.5$, $V_2 = 1.8$, $\omega_1 = 2.3$, $\omega_2 = 1.3$ and $k_B = 1$. The discussion is done by expressions (3.11) and (3.10). We then focus on the evolution of entanglement at equilibrium state, we specify the parameters β and κ_2.

We note that entanglement is more important for the couple electron-harmonic oscillator compared to the one electron, consequently the choice of the same reference scale κ, β, V_1 and V_2 shows that the particle number per unit volume, fully characterizes entanglement of the system at thermodynamic equilibrium.

4. System at Non-equilibrium State

In this section, we discuss the non-equilibrium state due to the dissipative mechanism of the system. It is characterized by the ordinary time-dependent density matrix solution of the following dissipative von Neumann equation:

$$i\hbar\frac{\partial}{\partial t}\rho(t) = \left[\hat{H}, \rho(t)\right] - \epsilon\left(\rho(t) - \rho_{eq}\right). \tag{35}$$

To show the potential applications from the TFD of entangled thermal state at non-equilibrium systems, the time-dependent density matrix is computed as follows

$$\rho_{neq}\left(\kappa_1, \kappa_2, \varepsilon_1, \varepsilon_2, \beta\right) = e^{-\epsilon t}U^+(t)\rho_0 U(t) + \left(1 - e^{-\epsilon t}\right)\rho_{eq}. \tag{36}$$

ρ_0 in (4.2) is the density matrix of the ground state and is written as

$$\rho_0 = \frac{\exp\left(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\right)\exp\left(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\right)}{Z\left(\kappa_1, \kappa_2, \varepsilon_1, \varepsilon_2, \beta\right)}|0, 0; 0_k\rangle\langle 0, 0; 0_k|, \tag{37}$$

where $U(t)$ is the unitary operator. It is reads

$$\begin{aligned}U(t) =& e^{iHt/\hbar} \\ =& \sum_{(n_1,n_2)=0}^{\infty} \int d^3k_3 \exp\left[i\frac{\hbar t}{2}\left(k_y^2 + (1-Q)k_z^2 + (1-S)k_x^2\right) \right. \\ & + it\left(\left(\gamma_1\varpi_1 n_1 + \gamma_2\varpi_2 n_2\right)\right. \\ & \left.\left. + \left(\gamma_1\varpi_1 + \gamma_2\varpi_2\right)\right)\right] |n_1, n_2; 1_{k_3'}\rangle\langle n_1, n_2; 1_{k_3'}|.\end{aligned} \tag{38}$$

Inserting (3.5), (4.3) and (4.4) into (4.2), we have

$$\begin{aligned}\rho_{neq}\left(\kappa_1, \kappa_2, \varepsilon_1, \varepsilon_2, \beta\right) =& \frac{1}{Z\left(\varepsilon_1, \varepsilon_2, K\right)} \sum_{(n_1,n_2=0)}^{\infty} \int d^3k_3 \left(\left(\delta_{n_1,0}\delta_{n_2,0}\delta_{1_k,0} - 1\right)e^{-\epsilon t} + 1\right) \\ & \times \exp\left[-\frac{\beta\hbar^2}{2}\left(k_y^2 + (1-Q)k_z^2 + (1-S)k_x^2\right)\right] \\ & \times \exp\left[-\beta\hbar\left(\gamma_1\varpi_1 n_1 + \gamma_2\varpi_2 n_2\right) - \frac{\beta\hbar}{2}\left(\gamma_1\varpi_1 + \gamma_2\varpi_2\right)\right] |n_1, n_2; 1_{k_3'}\rangle\langle n_1, n_2; 1_{k_3'}|.\end{aligned} \tag{39}$$

Going back to the formula $\rho = |\psi(t)\rangle\langle\psi(t)|$ and Eq (4.5), the extended density matrix becomes

$$\begin{aligned}\rho\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)=&\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\\&\times\sum_{(n_1,n_2)=0}^{\infty}\sum_{(m_1,m_2)=0}^{\infty}\int d^3k_3\int d^3k_3^{'}\sqrt{\big(\big(\delta_{n_1,0}\delta_{n_2,0}\delta_{1_{k_3},0}-1\big)e^{-\epsilon t}+1\big)}\\&\times\sqrt{\big(\big(\delta_{m_1,0}\delta_{m_2,0}\delta_{1_{{k_3}'},0}-1\big)e^{-\epsilon t}+1\big)}\exp\Bigg[-\frac{\beta\hbar^2}{4}\big(\big(k_{1y}^2+k_{2y}^2\big)+\big(1-Q\big)\big(k_{1z}^2+k_{2z}^2\big)\\&+\big(1-S\big)\big(k_{1x}^2+k_{2x}^2\big)\big)\Bigg]\exp\Bigg[-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big(m_1+n_1\big)-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big(m_2+n_2\big)\Bigg]\\&\times|m_1,m_2;1_{k_3}\rangle\langle n_1,n_2;1_{k_3'}||\tilde{m}_1,\tilde{m}_2;\tilde{1}_{k_3}\rangle\langle\tilde{n}_1,\tilde{n}_2;\tilde{1}_{k_3'}|.\end{aligned}\tag{40}$$

If we find the density matrix we can examine entanglement. This parameter is considered above, then the remainder is easy to handle by following the same procedures as the equilibrium case. We begin with:

4.1. Entanglement Between the Couple {1, 2} Oscillators and Electron {3}

We applied the same strategy as in the previous case for the non-dissipative mechanism, we obtain the extended density matrix as follows

$$\begin{aligned}\rho_{(3)}\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)=&\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)}\int d^3k_3\int d^3k_3^{'}\chi_{1_{k3},1_{{k_3}'}}(t)\\&\times\exp\Bigg[-\frac{\beta\hbar^2}{4}\Big(\big(k_{1y}^2+k_{2y}^2\big)+\big(1-Q\big)\big(k_{1z}^2+k_{2z}^2\big)+\big(1-S\big)\big(k_{1x}^2+k_{2x}^2\big)\Big)\Bigg]\\&\times|1_{k_3}\rangle\langle1_{k_3'}||\tilde{1}_{k_3}\rangle\langle\tilde{1}_{k_3'}|,\end{aligned}\tag{41}$$

where

$$\begin{aligned}\chi_{1_{k3},1_{{k_3}'}}(t)=&\Bigg(\sqrt{\big(\big(\delta_{1_{k_3},0}-1\big)e^{-\epsilon t}+1\big)}\sqrt{\big(\big(\delta_{1_{{k_3}'},0}-1\big)e^{-\epsilon t}+1\big)}\\&+\big(1-e^{-\epsilon t}\big)\frac{\exp\big(-\beta\hbar\gamma_1\varpi_1\big)\exp\big(-\beta\hbar\gamma_2\varpi_2\big)}{\big(1-\exp\big(-\hbar\beta\gamma_1\varpi_1\big)\big)\big(1-\exp\big(-\hbar\beta\gamma_2\varpi_2\big)\big)}\Bigg).\end{aligned}\tag{42}$$

Using (4.7), entanglement entropy

$$S_3=-k_B\mathrm{Tr}_3\left[\rho_3\log\rho_3\right],\tag{43}$$

is described as

$$\begin{aligned}S_3=&-k_B\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\Bigg(4\alpha_0(t)\log\big(\alpha_0(t)\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\big)\\&+2\times2^{\frac{3}{2}}\frac{\big(\frac{2\pi}{\beta\hbar^2}\big)^{\frac{3}{2}}}{\sqrt{1-Q}\sqrt{1-S}}\beta_0(t)\Bigg(\log\big(\beta_0(t)\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\big)-3\frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}}\Bigg)\\&+2^3\frac{\big(\frac{2\pi}{\beta\hbar^2}\big)^3}{\big(1-Q\big)\big(1-S\big)}\gamma_0(t)\Bigg(\log\big(\gamma_0(t)\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\big)-6\frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}}\Bigg)\\&+2\times2^{\frac{3}{2}}\frac{\big(\frac{2\pi}{\beta\hbar^2}\big)^{\frac{3}{2}}}{\sqrt{1-Q}\sqrt{1-S}}\beta_0(t)\Bigg(\log\big(\beta_0(t)\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\big)-3\frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}}\Bigg)\Bigg).\end{aligned}\tag{44}$$

We note that

$$\alpha_0(t)=\Bigg(1+\big(1-e^{-\epsilon t}\big)\frac{\exp\big(-\beta\hbar\gamma_1\varpi_1\big)\exp\big(-\beta\hbar\gamma_2\varpi_2\big)}{\big(1-\exp\big(-\hbar\beta\gamma_1\varpi_1\big)\big)\big(1-\exp\big(-\hbar\beta\gamma_2\varpi_2\big)\big)}\Bigg),\tag{45}$$

$$\beta_0(t)=\Bigg(\sqrt{\big(1-e^{-\epsilon t}\big)}+\big(1-e^{-\epsilon t}\big)\frac{\exp\big(-\beta\hbar\gamma_1\varpi_1\big)\exp\big(-\beta\hbar\gamma_2\varpi_2\big)}{\big(1-\exp\big(-\hbar\beta\gamma_1\varpi_1\big)\big)\big(1-\exp\big(-\hbar\beta\gamma_2\varpi_2\big)\big)}\Bigg), \tag{46}$$

and

$$\gamma_0(t)=\Bigg(\big(1-e^{-\epsilon t}\big)+\big(1-e^{-\epsilon t}\big)\frac{\exp\big(-\beta\hbar\gamma_1\varpi_1\big)\exp\big(-\beta\hbar\gamma_2\varpi_2\big)}{\big(1-\exp\big(-\hbar\beta\gamma_1\varpi_1\big)\big)\big(1-\exp\big(-\hbar\beta\gamma_2\varpi_2\big)\big)}\Bigg). \tag{47}$$

4.2. Entanglement Between {1} Oscillator and {2, 3} Oscillator-electron

Through a similar treatment, we find the reduced density matrice of the couple oscillator-electron as

$$\begin{aligned}\rho_{(2,3)}\left(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\right)=&\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\\&\times\sum_{(n_2,m_2)=0}^{\infty}\int d^3k_3\int d^3k_3^{'}\upsilon_{m_2,n_2;1_{k3},1_{k3'}}(t)\exp\Bigg[-\frac{\beta\hbar^2}{4}\Bigg(\big(k_{1y}^2+k_{2y}^2\big)\\&+\big(1-Q\big)\big(k_{1z}^2+k_{2z}^2\big)+\big(1-S\big)\big(k_{1x}^2+k_{2x}^2\big)\Bigg)-\frac{\hbar\beta\gamma_2}{2}\varpi_2\big(m_2+n_2\big)\Bigg]\\&\times|m_2,;1_{k_1}\rangle\langle n_2;1_{k_1'}||\tilde{m}_2;\tilde{1}_{k_3}\rangle\langle\tilde{n}_2;\tilde{1}_{k_3'}|,\end{aligned} \tag{48}$$

where

$$\begin{aligned}\upsilon_{m_2,n_2;1_{k3},1_{k3'}}(t)=&\Bigg(\sqrt{\big(\big(\delta_{n_2,0}\delta_{1_{k3},0}-1\big)e^{-\epsilon t}+1\big)}\sqrt{\big(\big(\delta_{m_2,0}\delta_{1_{k3'},0}-1\big)e^{-\epsilon t}+1\big)}\\&+\big(1-e^{-\epsilon t}\big)\frac{\exp\big(-\hbar\beta\gamma_1\varpi_1\big)}{\big(1-\exp\big(-\beta\hbar\gamma_2\varpi_2\big)\big)}\Bigg).\end{aligned} \tag{49}$$

We have the corresponding extended entropy

$$S_{2,3}=-k_B\mathrm{Tr}_{2,3}\left[\rho_{2,3}\log\rho_{2,3}\right], \tag{50}$$

in the form

$$\begin{aligned}S_{2,3}=&-k_B\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\Bigg(4\alpha_1(t)\log\big(\alpha_1(t)\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\big)\\&+4\times2^{\frac{3}{2}}\frac{\big(\frac{2\pi}{\beta\hbar^2}\big)^{\frac{3}{2}}}{\sqrt{1-Q}\sqrt{1-S}}\frac{\exp\big(\beta\hbar\gamma_2\varpi_2\big)}{\big(\exp\big(\frac{\beta\hbar\gamma_2\varpi_2}{2}\big)-1\big)}\beta_1(t)\\&\times\Bigg(\log\big(\beta_1(t)\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\big)-3\frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}}-\frac{\beta\hbar\gamma_2\varpi_2}{4}\frac{1}{\big(\exp\big(\frac{\beta\hbar\gamma_2\varpi_2}{2}\big)-1\big)}\Bigg)\\&+2^3\frac{\big(\frac{2\pi}{\beta\hbar^2}\big)^3}{\big(1-Q\big)\big(1-S\big)}\gamma_1(t)\Bigg(\log\big(\gamma_1(t)\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\big)-6\frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}}\Bigg)\\&+2^2\frac{\exp\big(\beta\hbar\gamma\varpi_2\big)}{\big(\exp\big(\frac{\beta\hbar\gamma\varpi_2}{2}\big)-1\big)^2}\gamma_1(t)\Bigg(\log\big(\gamma_1(t)\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\big)-\frac{1}{\big(\exp\big(\frac{\beta\hbar\gamma\varpi_2}{2}\big)-1\big)^2}\Bigg)\\&+2^3\frac{\big(\frac{2\pi}{\beta\hbar^2}\big)^3}{\big(1-Q\big)\big(1-S\big)}\frac{\exp\big(\beta\hbar\gamma\varpi_2\big)}{\big(\exp\big(\frac{\beta\hbar\gamma\varpi_2}{2}\big)-1\big)^2}\gamma_1(t)\\&\times\Bigg(\log\big(\gamma_1(t)\frac{\exp\big(-\frac{\beta\hbar\gamma_1}{2}\varpi_1\big)\exp\big(-\frac{\beta\hbar\gamma_2}{2}\varpi_2\big)}{Z\big(\kappa_1,\kappa_2,\varepsilon_1,\varepsilon_2,\beta\big)}\big)-6\frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}}-\frac{\beta\hbar\gamma\varpi_2}{\big(\exp\big(\frac{\beta\hbar\gamma\varpi_2}{2}\big)-1\big)}\Bigg)\Bigg),\end{aligned} \tag{51}$$

with $\alpha_1(t)$, $\beta_1(t)$, $\gamma_1(t)$ are given by

$$\alpha_1(t) = \left(1 + \left(1 - e^{-\epsilon t}\right)\frac{\exp\left(-\hbar\beta\gamma_1\varpi_1\right)}{\left(1 - \exp\left(-\beta\hbar\gamma_2\varpi_2\right)\right)}\right), \tag{52}$$

$$\beta_1(t) = \left(\sqrt{\left(1 - e^{-\epsilon t}\right)} + \left(1 - e^{-\epsilon t}\right)\frac{\exp\left(-\hbar\beta\gamma_1\varpi_1\right)}{\left(1 - \exp\left(-\beta\hbar\gamma_2\varpi_2\right)\right)}\right), \tag{53}$$

and

$$\gamma_1(t) = \left(\left(1 - e^{-\epsilon t}\right) + \left(1 - e^{-\epsilon t}\right)\frac{\exp\left(-\hbar\beta\gamma_1\varpi_1\right)}{\left(1 - \exp\left(-\beta\hbar\gamma_2\varpi_2\right)\right)}\right). \tag{54}$$

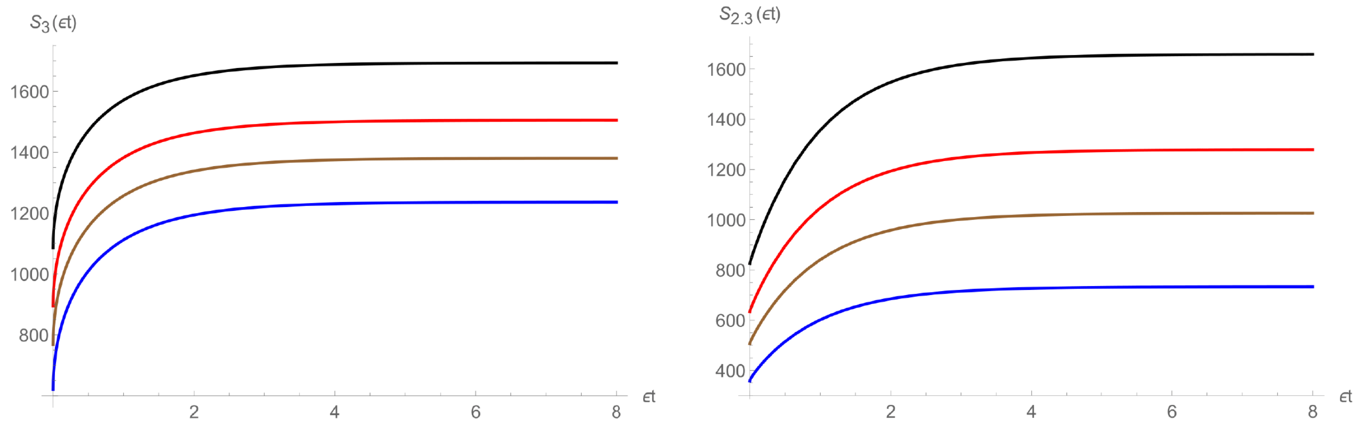

***Figure 2.** The time dependance of the extended entropies in Eq* (4.10) *and* (4.17) *for* $\beta = 0.5$ *and different values of* κ_2*:* $\kappa_2 = 1.8$ *(blue solid line),* $\kappa_2 = 2.5$ *(brown solid line),* $\kappa_2 = 3.1$ *(red solid line),* $\kappa_2 = 4$ *(black solid line).*

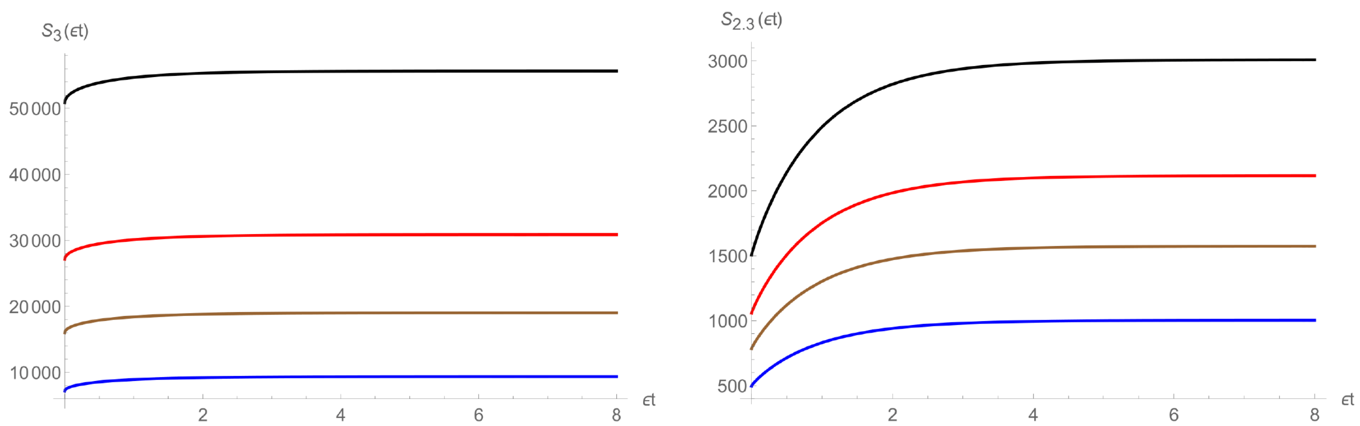

***Figure 3.** The time dependance of the extended entropies in Eq* (4.10) *and* (4.17) *for* $\kappa_2 = 0.5$ *and different values of* β*:* $\beta = 1.8$ *(blue solid line),* $\beta = 2.5$ *(brown solid line),* $\beta = 3.1$ *(red solid line),* $\beta = 4$ *(black solid line).*

With respect Figure 2 and by considering the case of one electron, entanglement is more larger for the small values of the κ_2 parameter ($\kappa_2 = 1.8$, 2.5 and 3.1). By increasing κ_2 particularly in the region $\epsilon t \geq 2$, entanglement evolution becomes equal between electron-harmonic oscillator and one electron. Saw this development, we expect that the trend will be reversed.

Compared with Fig1, we can conclude that from the Fig2, the system starts from a non-equilibrium state, it evolves towards the equilibrium state. We say that is an equilibrium with respect to κ_2 (relative equilibrium) but this is not a positional equilibrium because from Fig3, by following the increase of β, entanglement is more important and it reaches very large values of the case one electron.

5. Conclusion

Quantum entanglement of electron in interaction with two-mode electromagnetic field is studied using the TFD method. We derive the extended entropies by considering the cases equilibruim and non-equilibruim thermodynamic state compute between concerned and not concerned oscillator-electron. The importance of particle number per unit volume at thermodynamic equilibruim reflects the significance of entanglement consequently, entanglement appears to be more important for the couple electron-oscillator. This trend reverses follow the non-equilibruim state effect of entanglement parameter to know us qualitatively the relative equilibrium state.

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